

The Analytic Quantum Information Manifold

R. F. Streater,
Dept. of Mathematics, King's College London,
Strand, WC2R 2LS

February 7, 2008

Abstract

Let $H \geq I$ be a self-adjoint operator on a Hilbert space, such that $e^{-\beta H}$ is of trace-class for some $\beta < 1$. Let V be a symmetric operator such that $\|V\|_\omega := \|RV\| < \infty$, where $R = H^{-1}$. We show that the partition function $\text{Tr} e^{-(H+\lambda V)}$ is analytic in λ in a hood of the origin in the sense of Fréchet, in the Banach space with norm $\|\bullet\|_\omega$. This is applied to the quantum information manifold defined by H .
Keywords: Fisher metric, Bogoliubov Kubo Mori Green function, Orlicz space.

1 Introduction

The information manifold in non-parametric estimation theory has been introduced and studied by Pistone and Sempi [20, 5]. For a finite number of parameters, the theory is described in the books [1, 9]. The quantum version has been introduced by Hasagawa [7, 8] and by Nagaoka [14], and by Petz [16, 17]. This study is strictly valid only if the Hilbert space \mathcal{H} of states is finite dimensional. It provides a geometrical picture, as well as rigour, to the theory of linear response [10, 13, 4, 12, 3, 22]. For $\dim \mathcal{H} = \infty$, but limited to bounded potentials, the theory has been developed by Araki [2] in the context of Tomita-Takesaki theory. Our more concrete Hilbert-space version of this goes as follows.

Denote by \mathcal{C}_1 the set of trace-class operators on \mathcal{H} and let $\Sigma \subseteq \mathcal{C}_1$ denote the set of density operators. Given $\rho_0 \in \Sigma$ let $H_0 = -\log \rho_0 + cI \geq I$ be a self-adjoint operator with domain $\mathcal{D}(H_0)$ such that

$$\rho_0 = Z_0^{-1} e^{-H_0}. \quad (1)$$

We recognise H_0 , which is unique up to the constant c , as the Hamiltonian, and Z_0 as the partition function, whose finiteness expresses thermodynamic stability. Let X be a symmetric bounded operator, so $X \in \mathcal{B}(\mathcal{H})$, and perturb H_0 to $H_X = H_0 + X$, with corresponding state $\rho_X = Z_X^{-1} \exp(-H_X)$ and free energy $\psi_X = \log Z_X$. Araki proves that ψ_X is an analytic functional with a convergent Taylor series, whose first few terms are

$$\nabla_Y^+ \psi_X = \rho_X \cdot Y := \text{Tr}(\rho_X Y) \quad (2)$$

$$\nabla_Y^+ \nabla_Z^+ \psi_X = \int_0^1 d\lambda \text{Tr} \left\{ \rho_X^\lambda \hat{Y} \rho_X^{\lambda'} \hat{Z} \right\} := g_X(Y, Z) \quad (3)$$

$$\begin{aligned} \nabla_Y^+ \nabla_Z^+ \nabla_W^+ \psi_X &= \int_0^1 \prod d\alpha_i \delta(\sum \alpha_i - 1) \text{Tr}(\rho_X^{\alpha_1} Y \rho_X^{\alpha_2} Z \rho_X^{\alpha_3} W)_c \\ &:= t_X(Y, Z, W). \end{aligned} \quad (4)$$

Here, ∇_Y^+ denotes the Gateaux derivative of ψ_X in the direction $Y \in \mathcal{B}(\mathcal{H})$, $\hat{Y} = Y - \rho_X \cdot Y$ is the centred observable (the so-called “score”), $g_X(Y, Z)$ is the Bogoliubov-Kubo-Mori (BKM) metric, and $\text{Tr}(\dots)_c$ denotes the cumulant of third order. We adopt the convention throughout this paper that if $\lambda \in [0, 1]$ then $\lambda' = 1 - \lambda$. The torsion, t , measures the failure of g to be invariant under (+1)-parallel transport.

In [23, 24, 15] we began a study of the unbounded case. Let \mathcal{C}_p , $0 < p < 1$ denote the set of compact maps $A : \mathcal{H} \rightarrow \mathcal{H}$ such that $|A|^p \in \mathcal{C}_1$, and put

$$\mathcal{C}_{<1} := \bigcup_{0 < p < 1} \mathcal{C}_p. \quad (5)$$

It is known [19] that if $p < q$ then $\mathcal{C}_p \subset \mathcal{C}_q$ and that \mathcal{C}_p is a complete linear space with quasinorm

$$\|A\|_p := [\text{Tr}(A^* A)^{p/2}]^{1/p}. \quad (6)$$

In [24] we took the underlying set of the quantum information manifold to be

$$\mathcal{M} = \mathcal{C}_{<1} \cap \Sigma. \quad (7)$$

\mathcal{M} has a natural affine structure, coming from the linear structure of each \mathcal{C}_p ; it is called the (-1) -affine structure. Thus if $\rho_i \in \mathcal{C}_{p_i} \cap \Sigma$, $i = 1, 2$ let $p = \max\{p_1, p_2\}$; then $\rho_i \in \mathcal{C}_p \cap \Sigma$, $i = 1, 2$. We define $\lambda \rho_1 + \lambda' \rho_2$, $0 \leq \lambda \leq 1$ as the usual sum of operators in \mathcal{C}_p . We see that \mathcal{M} is (-1) -convex.

There is a quantum analogue of the classical Orlicz space $L \log L$: let

$$\mathcal{C}_1 \log \mathcal{C}_1 := \{\rho \in \mathcal{C}_1 : S(\rho) := -\sum_{i=1}^{\infty} \lambda_i \log \lambda_i < \infty\} \quad (8)$$

where $\{\lambda_i\}$ are the singular numbers of ρ . Thus the set of normal states of finite entropy is $\mathcal{C}_1 \log \mathcal{C}_1 \cap \Sigma$. It is easy to show that $\mathcal{C}_{<1} \subset \mathcal{C}_1 \log \mathcal{C}_1$.

The topology given to \mathcal{M} in [24] is not that induced on it as a subset of \mathcal{C}_1 . Following an idea in [20] we constructed a hood of $\rho_0 \in \mathcal{M}$ consisting of all $\rho \in \mathcal{M}$ which are ‘connected to ρ_0 by a one-parameter exponential family of states’. In [24] this is the family

$$\rho_{\lambda X} = Z_{\lambda}^{-1} e^{-(H_0 + \lambda X + c(\lambda))} \in \mathcal{M}, \quad 0 \leq \lambda \leq 1, \quad (9)$$

where X is a symmetric quadratic form defined on $\mathcal{D}(H_0^{1/2})$ and form-bounded relative to the energy form

$$q_0(\phi, \psi) := \langle H_0^{1/2} \phi, H_0^{1/2} \psi \rangle. \quad (10)$$

The state $\rho_{\lambda X}$ does not depend on the choice of $c(\lambda)$ and we do this so that $H := H_0 + \lambda X + c(\lambda) \geq I$. We write $R = R_X := H^{-1}$ for the inverse of H at $\lambda = 1$. If $\rho_0 \in \mathcal{C}_{\beta_0}$, to make sense we need that the q_0 -bound of X be less than $1 - \beta_0$. We then showed that $\rho_X \in \mathcal{C}_{\beta_X}$ for some $\beta_X < 1$. The set of q_0 -bounded quadratic forms X is a Banach space \mathcal{T}_0 with norm

$$\|X\|_0 := \|R_0^{1/2} X R_0^{1/2}\|, \quad (11)$$

where $\|\bullet\|$ is the operator norm for \mathcal{H} , and $R_0 = H_0^{-1}$. We showed that sufficient for $\rho_X \in \mathcal{M}$ is that $\|X\|_0 < 1 - \beta_0$, and that Z_X is a Lipschitz function of X ; however, we could not show that the first moment $\text{Tr}(\rho_0 X)$ is finite. We introduced the regularised mean of $X \in \mathcal{T}_0$,

$$\rho_0.X := \text{Tr}[\rho_0^{\lambda} X \rho_0^{\lambda'}] \quad (12)$$

and showed that it is finite and independent of $\lambda \in (0, 1)$.

The hood of ρ_0 defined by $\{\rho_X : \|X\|_0 < 1 - \beta_0\}$ can be mapped bijectively onto the open ball of radius $1 - \beta_0$ in the Banach space

$$\hat{\mathcal{T}}_0 := \{X \in \mathcal{T}_0 : \rho_0.X = 0\}; \quad (13)$$

thus, $\hat{\mathcal{T}}_0$ consists of score variables. This map makes the hood into a manifold modelled on $\hat{\mathcal{T}}_0$. The linear structure of $\hat{\mathcal{T}}_0$ induces a local affine structure

on \mathcal{M} , called the (+1)-affine structure. To any point ρ_X , we may similarly define a hood of states of ρ_X to be

$$\left\{ \rho_Y : \|Y\|_X := \|R_X^{1/2} Y R_X^{1/2}\| < a \leq 1 - \beta_X \right\}, \quad (14)$$

which is in bijection with an open ball in the Banach space

$$\hat{\mathcal{T}}_X := \{Y : \|Y\|_X < \infty \text{ and } \rho_X \cdot Y = 0\}. \quad (15)$$

We showed that on the overlap region, the norms $\|\bullet\|_0$ and $\|\bullet\|_X$ are equivalent norms; the (+1)-affine structures on the region of overlap are also the same. Thus we have covered \mathcal{M} by a Banach manifold with a local affine structure. We showed that the connected component containing ρ_0 is (+1)-convex.

In the present paper we consider the problem of furnishing \mathcal{M} with a Riemannian metric. As shown by Petz [17], in the finite-dimensional case there are several candidates, but for geometric reasons we choose the *BKM* metric. Indeed, the studies of Nagaoka [14] show that +1 and -1-affine structures are not dual relative to the ‘Uhlmann’ metric, (the real part of $\text{Tr } \rho X^* Y$). They *are* dual, however, relative to the *BKM*-metric. Since this is the second derivative of ψ , in infinite dimensions we must impose more regularity than we have so far. Apart from having this ‘Amari’ duality, similar to the classical Fisher-Rao metric, the *BKM*-metric is the one entering Kubo’s theory of linear response, as well as the one arising in quantum statistical dynamics.

In this paper we assume that X is an operator which is H_0 -bounded. We then say that the tangent direction X is analytic. We show in §(2) that this leads to the Fréchet differentiability of ψ_0 in the analytic direction X , with Lipschitz continuous derivative

$$\nabla_X \psi_0 = \rho_0 \cdot X, \quad (16)$$

and that $\rho_0 X$ is of trace class. This means that the regularisation of the mean is not necessary. We show that a metric g can be defined by eq. (3).

In §(3) we show that the n -point Kubo cumulant is finite, and that for small $|\lambda|$ the Taylor series for $Z_{\lambda V}$ converges if V is an analytic direction. We show that when we furnish \mathcal{M} with the analytic topology, it becomes a real analytic Banach manifold.

2 H_0 -bounded perturbations

Let $H_0 \geq I$ be a selfadjoint operator with domain $\mathcal{D}(H_0)$ and quadratic form q_0 . A necessary and sufficient condition for a symmetric operator $X : \mathcal{D}(H_0) \rightarrow \mathcal{H}$ to be H_0 -bounded is that $R_0 X$ be bounded [21]. One can show that

$$\|X\|_0 := \|R_0^{1/2} X R_0^{1/2}\| \leq \|R_0 X\| \quad (17)$$

It follows that if X is H_0 -bounded, then its form is q_0 -bounded. Suppose now that $\rho_0 := Z_0^{-1} e^{-H_0} \in \mathcal{C}_{\beta_0}$, $\beta_0 < 1$, and that $\|R_0 X\| < 1 - \beta_0$. Then by eq. (17), $\|X\|_0 < 1 - \beta_0$ and so by [24] the perturbed state

$$\rho_X = Z_X^{-1} e^{-H_X} \in \mathcal{C}_{\beta_X} \subset \mathcal{C}_{<1} \quad (18)$$

lies in a hood of ρ_0 in the topology induced by \mathcal{T}_0 . Let us define a stronger topology than this, by taking hoods of ρ_0 to be

$$\{\rho_X : \|X\|_\omega := \|R_0 X\| < a \leq 1 - \beta_0\}. \quad (19)$$

We call this the analytic or ω -topology, and $\|\bullet\|_\omega$ the ω -norm. Similarly, if ρ_X is in the analytic hood of ρ_0 , and Y is H_X -bounded with $\|R_X Y\| < 1 - \beta_X$, then $\rho_{X+Y} \in \mathcal{M}$ and the analytic hoods of ρ_X can be defined by

$$\{\rho_{X+Y} : \|R_X Y\| < a \leq 1 - \beta_X\}. \quad (20)$$

Let $\hat{\mathcal{T}}_\omega(\rho_X) := \{Y : \|Y R_X\| < \infty, \rho_X \cdot Y = 0\}$. Then the construction of the ω -hoods covers \mathcal{M} by sets homeomorphic to balls in the Banach spaces $\hat{\mathcal{T}}_\omega(\rho)$; these have mutually equivalent norms on the overlaps of the hoods. To see this, suppose that Y is in the hood of ρ_0 and also of ρ_X . The charts around ρ_0 and ρ_X take ρ_Y to balls in the Banach spaces $\hat{\mathcal{T}}_\omega(\rho_0)$ and $\hat{\mathcal{T}}_\omega(\rho_X)$, with norms $\|R_0 Y\|$ and $\|R_X Y\|$. These are equivalent, since

$$m \|Y R_0\| \leq \|Y R_X\| \leq M \|R_0 Y\| \quad \text{for all } Y, \quad (21)$$

where $m = \|(H_X + I)R_0\|^{-1}$ and $M = \|(H_0 + I)R_X\|$. Thus \mathcal{M} is made into a Banach manifold, called the ω quantum information manifold. This name will be justified in the next section. It is easy to see that the local (+1)-affine structure coming from the linear structure of $\hat{\mathcal{T}}_\omega(\rho_X)$ is compatible with that given in [24].

The following lemma will be needed later. It is clear that R_{X+Y} maps \mathcal{H} into $\mathcal{D}(H_0) = \mathcal{D}(H_X) = \mathcal{D}(H_{X+Y})$. It follows from lemma 6, [24] that $H_X R_{X+Y}$ is bounded. We find

$$(I + Y R_X)(H_X R_{X+Y}) = H_X R_{X+Y} + Y R_{X+Y} = I. \quad (22)$$

Hence $H_X R_{X+Y} = (I + Y R_X)^{-1}$, and we get if $\|Y R_X\| < 1$,

$$\|H_X R_{X+Y}\| \leq (1 - \|Y R_X\|)^{-1}. \quad (23)$$

From this we see

Lemma 2.1 *If $a < 1$ and X is H_0 -small, then $\|H_X R_{X+Y}\|$ is bounded in the set $\{Y : \|Y R_X\| \leq a\}$. \square*

Now suppose that X perturbs H_0 and Y perturbs H_X , as above. Then

Theorem 2.2 *If $1 - 2\delta > \beta_X$, then $\rho_X^{\delta'} Y$ has a trace-class extension.*

Proof. Choose $\delta > 0$ so that $\rho_X^{1-2\delta} \in \mathcal{M}$. Then

$$\begin{aligned} \|\rho_X^{1-\delta} Y\|_1 &= \|Z_X^{\delta-1} e^{-(1-\delta)H_X} Y\|_1 \\ &\leq C \|\rho_X^{1-2\delta}\|_1 \|e^{-\delta H_X} H_X\|_\infty \|R_X Y\|_\infty \\ &< \infty. \end{aligned}$$

\square

Corollary. The regularised mean coincides with the true mean:

$$\text{Tr}(\rho_X^\lambda Y \rho_X^{1-\lambda}) = \text{Tr}(\rho_X Y), \quad \text{for all } \lambda \in (0, 1). \quad (24)$$

For, we can write $\rho_X Y = \rho_X^\delta \rho_X^{\delta'} Y$ and use the cyclicity of the trace.

Theorem 2.3 *In the analytic manifold, $\psi_X := \log Z_X$ is ∇_Y^+ Fréchet differentiable, and*

$$\nabla_Y^+ \psi_X = \rho_X \cdot Y \quad (25)$$

Proof. By two applications of Duhamel's formula [24], Th. 9, we get for the difference quotient,

$$\begin{aligned} &\lambda^{-1} \text{Tr} \left[e^{-(H_X + \lambda Y)} - e^{-H_X} \right] - \text{Tr} \left[e^{-H_X} Y \right] = \\ &= \text{Tr} \int_0^1 \alpha d\alpha \int_0^1 d\beta \left\{ e^{-\alpha\beta(H_X + \lambda Y)} \lambda Y e^{-\alpha\beta' H_X} Y e^{-\alpha' H_X} \right\}. \end{aligned}$$

We can put the trace inside the integral, by Fubini's theorem, if we can show that by doing so we get an absolutely convergent integral of the trace norm. The right hand side is then obviously $O(\lambda)$, with constant equal to the trace of an operator of the form

$$C = \int \int \int_0^1 \rho_1^{\alpha_1} Y \rho_2^{\alpha_2} Z \rho_3^{\alpha_3} \prod_{i=1}^3 d\alpha_i \delta \left(\sum_{i=1}^3 \alpha_i - 1 \right), \quad (26)$$

which is the quantum version of the integral remainder in Taylor's theorem.

We divide the region of integration into three (overlapping) parts; in region i , $\alpha_i \geq 1/3$, ($i = 1, 2, 3$). The estimate is similar in each case. The hardest case is when the variable larger than $1/3$, say α_1 , involves the state $\rho_1 = Z_\lambda^{-1} e^{-(H_X + \lambda X)}$, as this requires an estimate independent of λ . We therefore do this case in detail. So take $\alpha_1 \geq 1/3$, and $\alpha_2, \alpha_3 > 0$, and $H_i \geq I$. Put $R_i = H_i^{-1}$. We now show that for small $\delta > 0$, $\rho_1^{\alpha_1 - \delta} Y \rho_2^{\alpha_2} Z \rho_3^{\alpha_3}$ is of trace class. Indeed, by Hölder,

$$\begin{aligned} \|\rho_1^{\alpha_1 - \delta} Y \rho_2^{\alpha_2} Z \rho_3^{\alpha_3}\|_1 &\leq \|\rho_1^{\alpha_1 - 2\delta}\|_{1/\alpha_1} \|Y \rho_1^\delta\|_\infty \|\rho_2^{\alpha_2}\|_{1/\alpha_2} \\ &\quad \|Z \rho_3^\gamma\|_\infty \|\rho_3^{\alpha_3 - \gamma}\|_{1/\alpha_3} \\ &\leq \|\rho_1^{1 - 2\delta/\alpha_1}\|_1^{\alpha_1} \|Y R_1\|_\infty \|H_1 \rho_1^\delta\|_\infty \\ &\quad \|Z R_3\|_\infty \|H_3 \rho_3^\gamma\|_\infty \|\rho_3^{1 - \gamma/\alpha_3}\|_1^{\alpha_3}. \end{aligned}$$

If $2\delta < \beta'_1 \alpha_1$ and $\gamma < \alpha_3 \beta'_3$, then all the norms are finite; here, the β_i are such that $\rho_i \in \mathcal{C}_{\beta_i}$. The size of γ depends on α_3 , but in the region $\alpha_1 \geq 1/3$, δ can be chosen independent of α_i . Now take $2\delta < \beta'_1/3$. Then for each $\alpha_1 \geq 1/3$, $\alpha_2 > 0$, $\alpha_3 > 0$ we can take a bit, ρ_1^δ , of the dominant factor to the right end. We get, using $\sum \alpha_i = 1$ and Hölder,

$$\begin{aligned} \|C\|_1 &= \|\rho_1^{\alpha_1 - 2\delta} (\rho_1^\delta Y) (\rho_2^{\alpha_2}) (Z R_3) \rho_3^{\alpha_3} (H_3 \rho_1^\delta)\|_1 \\ &\leq \|\rho_1^{\alpha_1 - 2\delta}\|_{1/\alpha_1} \|\rho_1^\delta Y\|_\infty \|\rho_2^{\alpha_2}\|_{1/\alpha_2}^{\alpha_2} \|Z R_3\|_\infty \\ &\quad \|\rho_3^{\alpha_3}\|_{1/\alpha_3}^{\alpha_3} \|H_3 R_1\|_\infty \|H_1 \rho_1^\delta\|_\infty. \end{aligned}$$

The first norm is bounded by

$$\|\rho_1^{1 - 2\delta/\alpha_1}\|_1^{\alpha_1} \leq \|\rho_1^{1 - 6\delta}\|_1 < \infty$$

as $1 - 6\delta > \beta_1$. The second factor,

$$\|\rho_1^\delta Y\|_\infty \leq \|\rho_1^\delta H_1\|_\infty \|R_1 Y\|_\infty$$

is finite, independent of λ , by the spectral theorem, and by lemma (2.1). The same can be said of the factors $\|H_1 \rho_1^\delta\|_\infty$ and $\|H_3 R_1\|_\infty$. The factors involving ρ_2 and ρ_3 are unity, as they are states. Hence the trace norm of C is bounded (in the region 1) independent of α and λ , and its integral is bounded. The other regions, 2 and 3, are treated similarly. \square

Corollary. The *BKM* metric

$$g_X(Y, Z) = \int_0^1 d\alpha \text{Tr} \left(\rho_X^\alpha Y \rho_X^{1-\alpha} Z \right) \quad (27)$$

is finite if Y, Z are ω -directions. Moreover, ψ_X is Fréchet differentiable in the Y direction in the analytic manifold, with derivative $\rho_X.Y$; for

$$|\rho_X.Y| = |\text{Tr}[\rho_X^{1-\delta}(\rho_X H_X)(R_X Y)]| \leq C \|R_X Y\| \quad (28)$$

is continuous in the ω -norm

3 The analyticity of the free energy

In this section, we show that the free energy $\psi_{\lambda X}$ is an analytic function in a hood of $\lambda = 0$ if X is an ω -direction. Suppose $\rho = Z^{-1} \exp(-H) \in \mathcal{C}_\beta \in \mathcal{M}$, and V_j are H -bounded, $j = 1, \dots, n$. As before, we assume that $H \geq 1$ and write $R = H^{-1}$.

We start with the n -point function

$$M_n := Z_0 \text{Tr} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \dots \int_0^1 d\alpha_{n-1} [\rho^{\alpha_1} V_1 \rho^{\alpha_2} V_2 \dots \rho^{\alpha_n} V_n].$$

Here, $\alpha_n = 1 - \alpha_1 - \dots - \alpha_{n-1}$. Write this trace as

$$\begin{aligned} & \left[\rho^{\alpha_1 \beta} \right] \left[H^{1-\delta_n+\delta_1} \rho^{(1-\beta)\alpha_1} \right] \left[R^{\delta_1} V_1 R^{1-\delta_1} \right] \left[\rho^{\alpha_2 \beta} \right] \left[H^{1-\delta_1+\delta_2} \rho^{(1-\beta)\alpha_2} \right] \\ & \left[R^{\delta_2} V_2 R^{1-\delta_2} \right] \dots \left[\rho^{\alpha_n \beta} \right] \left[H^{1-\delta_{n-1}+\delta_n} \rho^{(1-\beta)\alpha_n} \right] \left[R^{\delta_n} V_n R^{1-\delta_n} \right], \end{aligned}$$

where $\delta_j \in (0, 1)$ are yet to be chosen. This is the product of n factors of the form $\left[\rho^{\alpha_j \beta} \right]$, n factors of the form $\left[H^{1-\delta_{j-1}+\delta_j} \rho^{(1-\beta)\alpha_j} \right]$, where δ_0 means δ_n , and n factors of the form $\left[R^{\delta_j} V_j R^{1-\delta_j} \right]$. By interpolation, the last type is bounded in operator norm by [21]

$$\|R^{\delta_j} V_j R^{1-\delta_j}\| \leq \|R V_j\| = \|V_j\|_\omega.$$

This bound is independent of α . The factors $\rho^{\alpha_j \beta}$ are bounded in trace-norm by the Hölder inequality

$$\|A_1 \dots A_n\|_1 \leq \|A_1\|_{p_1}^{1/p_1} \dots \|A_n\|_{p_n}^{1/p_n}.$$

Since $\sum \alpha_j = 1$ we may put $p_j = 1/\alpha_j$ to get

$$\begin{aligned} & \left\| \left[\rho^{\alpha_1 \beta} \right] \dots \left[\rho^{\alpha_n \beta} \right] \right\|_1 \leq \|\rho^\beta\|_1^{\alpha_1} \dots \|\rho^\beta\|_1^{\alpha_n} \\ & = \|\rho^\beta\|_1 < \infty. \end{aligned}$$

This bound is also independent of the α' s. We bound the remaining factors in operator norm. By the spectral theorem we have the bound

$$\begin{aligned} \|\rho^{(1-\beta)\alpha_j} H^{1-\delta_{j-1}+\delta_j}\|_\infty &= Z^{-\alpha_j(1-\beta)} \sup_{x \geq 1} \{e^{-(1-\beta)\alpha_j x} x^{1-\delta_{j-1}+\delta_j}\} \quad (29) \\ &\leq Z^{-\alpha_j(1-\beta)} \left(\frac{1-\delta_{j-1}+\delta_j}{(1-\beta)\alpha_j} \right)^{1-\delta_{j-1}+\delta_j} e^{-(1-\delta_{j-1}+\delta_j)}. \quad (30) \end{aligned}$$

We have to integrate $\alpha_j^{-(1-\delta_{j-1}+\delta_j)} d\alpha_j$; the region of integration is the union of n (overlapping) regions $S_j := \{\alpha : \alpha_j \geq 1/n\}$. We treat each in a similar manner. For S_n we can ensure integrability at $\alpha_j = 0$, $j = 1, \dots, n-1$ if we choose δ_j so that $1-\delta_{j-1}+\delta_j < 1$, that is, $\delta_j < \delta_{j-1}$. So we choose $\delta_n = \delta_0 > \delta_1 > \delta_2 > \dots > \delta_{n-1}$. To ensure that $\delta_j \in [0, 1]$ it is enough to choose $\delta_n = 1$, $\delta_1 = 1 - 1/n$, \dots , $\delta_{n-1} = 1/n$. Then each of the $n-1$ integrals for $j = 1, \dots, n-1$,

$$\int_0^1 d\alpha_j \alpha_j^{-(1-\delta_{j-1}+\delta_j)} = (\delta_{j-1} - \delta_j)^{-1}$$

is equal to n , giving a factor n^{n-1} . The remaining α factor in the region S_n is $\alpha_n^{-(1-\delta_{n-1}+\delta_n)} \leq n^2$. We get the same bound in the other regions, S_j $j = 1, \dots, n-1$, so we get the bound $n^3 n^{n-1} = n^2 n^n$. The other factors in the estimate (30) can be bounded independent of α :

$$\begin{aligned} &\prod_{j=1}^n Z^{-\alpha_j(1-\beta)} (1-\delta_{j-1}+\delta_j)^{1-\delta_{j-1}+\delta_j} (1-\beta)^{-(1-\delta_{j-1}+\delta_j)} e^{-(1-\delta_{j-1}+\delta_j)} \\ &\leq 4Z^{-(1-\beta)} (1-\beta)^{-n} e^{-n} \end{aligned}$$

since $(1-\delta_{j-1}+\delta_j) < 1$ except for one term, when it is less than 2. Thus we get for the n -point function the bound

$$4\|\rho^\beta\|_1 Z^{-\beta} n^2 n^n e^{-n} \prod_j \|V_j/(1-\beta)\|_\omega.$$

It follows that the partition function, $Z_{\lambda V}$, has a convergent Taylor series $\sum \lambda^n M_n/n!$ if $|\lambda| \|V\|_\omega < 1-\beta$. Recall that this is also the condition that $\rho_{\lambda V}$ lie in \mathcal{M} in an ω -hood of ρ . Since $Z_{\lambda V} > 0$, it follows that the free energy, $\psi := \log Z_{\lambda V}$, is real analytic in λ in the same interval.

4 Conclusion

Suppose that $\beta < 1$. We have seen that the condition that the perturbation of the Hamiltonian $H \geq I$ of a state $\rho \in \mathcal{C}_\beta$ by a small enough symmetric operator V leads to another state $\rho_V \in \mathcal{M}$. We defined a topology on \mathcal{M} in a hood of ρ using the norm $\|V\|_\omega(\rho) = \|RV\|$, where $RH = I$; this hood \mathcal{U} is set of states ρ_V coming from the ball of operators V with $\|V\|_\omega < 1 - \beta$. This is finer than the topology defined by the norm $\|V\| = \|R^{1/2}VR^{1/2}\|$ which is finite for all form-bounded perturbations. We saw that the Taylor series for the partition function converges in the same hood of ρ . The ω -norms associated with different points of \mathcal{M} are equivalent on the overlap of two hoods, so together we have a Banach manifold modelled on the Banach spaces of H -bounded symmetric operators. The question arises, is there an analytic structure of this manifold? In fact, an analytic structure is deemed to be provided if we specify the ring of germs of analytic functions at each point. Let us say that a map $\psi : \mathcal{U} \rightarrow \mathbf{C}$ is $+1$ -analytic if in \mathcal{U} it is infinitely often Fréchet differentiable and $\psi(\rho_{\lambda V})$ has a convergent Taylor expansion in λ , for all ω -directions V in the tangent space. Because of the equivalence of norms, this concept is coordinate-free. Thus an analytic structure for \mathcal{M} has been specified. In particular, the mixture coordinates $\eta_X = \rho.X$ are analytic, as they are derivatives of the free energy. In his theory of expansions, Araki [2] showed that the free energy ψ_X is an entire function in the Banach space $\mathcal{B}(\mathcal{H})$ if the perturbation X is bounded, in the more general context of Tomita-Takesaki theory. With unbounded perturbations we cannot hope for entire functions; for we need only take $V = -H_0$ to hit a singularity in ψ_X . However, it is likely that some analyticity remains, even in the thermodynamic limit, for relatively bounded perturbations.

Compared with [23], we have improved the result in several ways. We show analyticity instead of twice-differentiability; we have dropped the commutator condition altogether; the manifold is infinite dimensional instead of finite-dimensional; and we have enlarged the class of states to $\mathcal{C}_{<1}$. Further results are obtained in [6].

Acknowledgements This continues work started at CNRS Luminy, Marseille; I thank P. Combe for arranging the visit, and CNRS for financial support. Thanks are due to P. Combe, G. Burdet and H. Nencka for useful discussions.

References

- [1] S.-i. Amari, *Differential Geometric Methods in Statistics*, **Lecture Notes in Statistics**, **28**, 1985. Springer-Verlag.
- [2] Araki, H., Publ. R. I. M. S. (Kyoto), **9**, 165-209, 1968.
- [3] Balian, R., Y. Alhassid, H. Reinhardt, *Dissipation in many-body systems: a geometrical approach based on information theory*, Physics Reports, **131**, 1-146, 1986. North Holland.
- [4] Bogoliubov, N. N., Phys. Abh. Sov. Union, **1**, 229-, 1962.
- [5] Gibilisco, P., and G. Pistone, *Connections on nonparametric statistical manifolds by Orlicz space geometry*, Infinite-dimensional Analysis, Quantum Probability and Related Topics, **1**, 325-347, 1998.
- [6] Grasselli, M. R., and R. F. Streater, *The quantum information manifold for epsilon-bounded forms*, math-ph/9910031.
- [7] Hasagawa, H. Reports on Math. Phys., **33**, 87, 1993.
- [8] Hasagawa, H. *Noncommutative Extension of the Information Geometry*, in **Quantum Communication and Measurement**, Eds V. P. Belavkin, O. Hirota and R. L. Hudson, 1995, Plenum Press, pp 327-337,
- [9] Kass, R. A. and P. W. Vos, "Geometric Foundations of Asymptotic Inference", Wiley, NY 1997.
- [10] Kubo, R. Reports on Progress in Physics, **29**, 255-284, 1966.
- [11] Lesniewski, A., and M. B. Ruskai, *Relative Entropy and Monotone Riemannian Metrics on Noncommutative Probability Spaces*, preprint.
- [12] Matsubara, T., Prog. Theor. Phys., **14**, 351-, 1955.
- [13] Mori, H., Prog. Theor. Phys., **33**, 423-, 1965.
- [14] Nagaoka, H., pp 449-452 in **Quantum Communication and Measurement**, Eds V. P. Belavkin, O. Hirota and R. L. Hudson, Plenum Press, New York, 1995.
- [15] H. Nencka and R. F. Streater, *Information Geometry for Some Lie Algebras*, to appear in Infinite Dimensional Analysis and Quantum Probability, World Scientific.

- [16] Petz, D., and Toth, G. Lett. Math. Phys., **27**, 205-216, 1993.
- [17] Petz, D., *Monotone Metrics on Matrix Spaces*, Linear Alg. and Appl., **244**, 81-96, 1996.
- [18] Petz, D. and C. Sudar, *Geometries of Quantum States*, Journ. Math. Phys., **37**, 2662-73, 1996.
- [19] Pietsch, A. **Nuclear Locally Convex Spaces**, Springer-Verlag, 1972.
- [20] Pistone, G. and C. Sempi, Annals of Statistics, **23**, 1543-1561, 1995.
- [21] Reed, M. and B. Simon, **Methods of Modern Mathematical Physics**, Academic Press, Vol. 2, 1975.
- [22] Roepstorff, G. Commun. Math. Phys., **46**, 253-262 1976.
- [23] R. F. Streater, *Information Geometry and Reduced Quantum Description*, Reports on Mathematical Physics, **38**, 419-436 1996.
- [24] R. F. Streater, *The Information Manifold for Relatively Bounded Potentials*, to appear in the Bogoliubov 90th Birthday Memorial Volume. Ed A. A. Slavnov. math-ph/9910035